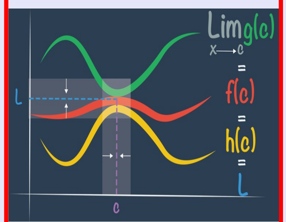
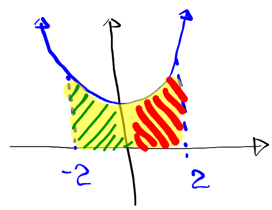
Math 261

Spring 2022

Lecture 23

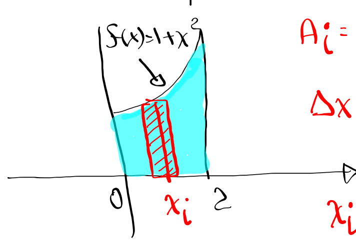


Find the area below $f(x) = 1 + x^2$, above x-axis for $-2 \le x \le 2$.



$f(-x) = 1 + (-x)^2 = 1 + x^2 = f(x)$

Symmetric with respect to Y-axis.



$A_i = \Delta x \cdot f(x_i)$

$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_i = a + i\,\Delta x = 0 + i \cdot \frac{2}{n}$

$x_i = \frac{2i}{n}$

$f(x_i) = 1 + x_i^2 = 1 + \left(\frac{2i}{n}\right)^2$

$f(x_i) = 1 + \frac{4i^2}{n^2}$

$A_i = f(x_i) \cdot \Delta x$

$$\boxed{A_i = \left(1 + \frac{4i^2}{n^2}\right) \cdot \frac{2}{n}}$$

$$A = 2\lim_{n\to\infty} \sum_{i=1}^{n} A_i$$

$$= 2\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{4i^2}{n^2}\right)\cdot\frac{2}{n}$$

$$= 2\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{4i^2}{n^2}\right)$$

$$= 2\lim_{n\to\infty} \frac{2}{n}\left[\sum_{i=1}^{n} 1 + \sum_{i=1}^{n} \frac{4i^2}{n^2}\right]$$

$$= 2\lim_{n\to\infty} \frac{2}{n}\left[n + \frac{4}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= 2\lim_{n\to\infty} \frac{2}{n}\left[n + \frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 2\lim_{n\to\infty} \left[2 + \frac{\overset{16n^3+\cdots}{8n(n+1)(2n+1)}}{6n^3}\right]$$

$$= 2\cdot\left[2 + \frac{16}{6}\right] = 2\left(2 + \frac{8}{3}\right)$$

$$= 2\cdot\frac{14}{3} = \boxed{\frac{28}{3}}$$

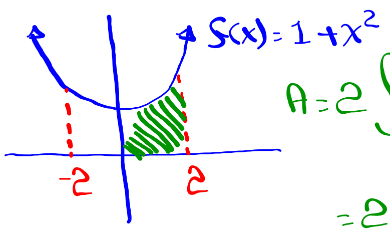
Definition of Definite integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \int_a^b f(x)\,dx = F(x)\Big|_a^b$$

where $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\cdot\Delta x$

$$F'(x) = f(x)$$

f(x) must be continuous on [a,b]



$$A = 2\int_0^2 (1 + x^2)\,dx$$

$$= 2\left[x + \frac{x^3}{3}\right]\Big|_0^2$$

$$= 2\left[2 + \frac{2^3}{3} - 0 - \frac{0^3}{3}\right] = 2\left[2 + \frac{8}{3}\right] = 2\cdot\frac{14}{3} = \boxed{\frac{28}{3}}$$

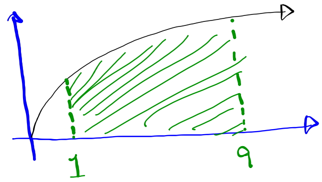
Suppose $f(x) \geq 0$, continuous on $[a,b]$

Area below $f(x)$, above x-axis on $[a,b]$ is

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a) \quad \text{where}$$

$$F'(x) = f(x).$$

Find the area below $f(x) = \sqrt{x}$, above x-axis from $x=1$ to $x=9$.



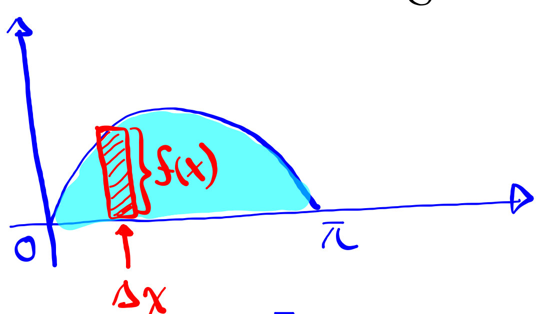
$$A = \int_1^9 \sqrt{x}\,dx = \int_1^9 x^{\frac{1}{2}}\,dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Big|_1^9 = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_1^9 = \frac{2}{3}x\sqrt{x}\Big|_1^9$$

$$= \frac{2}{3}\left[9\sqrt{9} - 1\sqrt{1}\right] = \frac{2}{3}\left[27-1\right] = \frac{2}{3}\cdot 26$$

$$= \boxed{\frac{52}{3}}$$

Find the area bounded $f(x) = \sin x$, $\overset{\text{x-axis}}{y=0}$, $x=0$, and $x=\pi$.



$$A = \int_0^\pi f(x)\,dx = \int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi$$

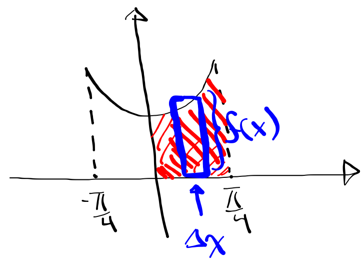
$$= -\left[\overset{-1}{\cancel{\cos\pi}} - \overset{1}{\cancel{\cos 0}}\right]$$

$$= -\left[-1 - 1\right] = \boxed{2}$$

Find the area below $f(x) = \sec^2 x$, above x-axis on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

$f(x) = \sec^2 x$ cont. on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, $f(x) \geq 0$

$f(-x) = f(x)$
even function
Y-axis



$$A = 2\int_0^{\pi/4} f(x)\,dx$$

$$= 2\int_0^{\pi/4} \sec^2 x\,dx$$

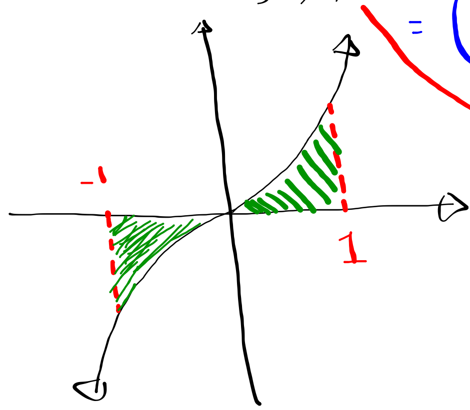
$$= 2\left[\tan x\right]\Big|_0^{\pi/4}$$

$$= 2\left[\underbrace{\tan \tfrac{\pi}{4}}_{1} - \underbrace{\tan 0}_{0}\right]$$

$$= 2[1 - 0] = \boxed{2}$$

Evaluate $\displaystyle\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1}$

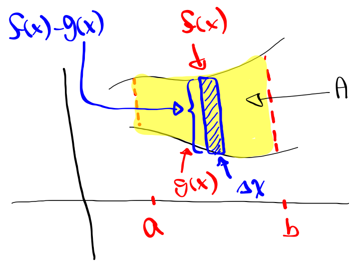
$f(x) = x^3$



$$= \left(\frac{1^4}{4} - \frac{(-1)^4}{4}\right) = \frac{1}{4} - \frac{1}{4} = \boxed{0}$$

odd function
$f(-x) = -f(x)$
Symmetry: Origin

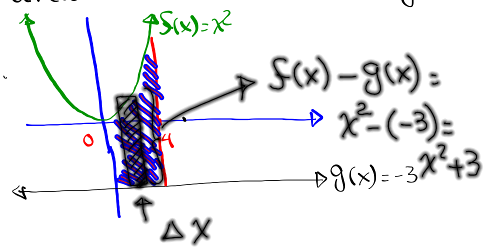
How to find the area between $f(x)$ and $g(x)$ on $[a,b]$ where $f(x) \geq g(x)$:



Top Bottom

$$A = \int_a^b [f(x) - g(x)]\,dx$$

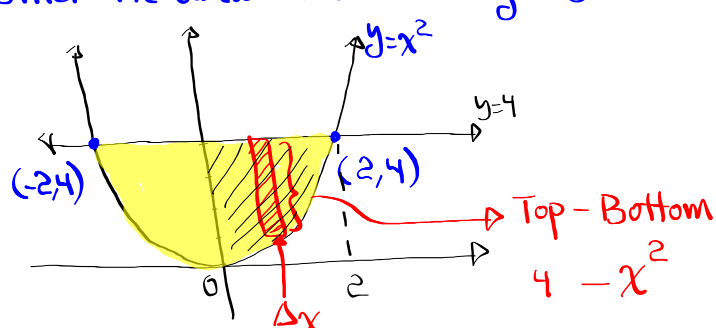
Find the area between $f(x) = x^2$ and $g(x) = -3$ on $[0,4]$.



$$A = \int_0^4 [x^2+3]\,dx = \left(\frac{x^3}{3} + 3x\right)\Big|_0^4 = \frac{4^3}{3} + 3(4) - 0$$

$$= \frac{64}{3} + 12 = \boxed{\frac{100}{3}}$$

Find the area bounded by $y=4$ and $y=x^2$.



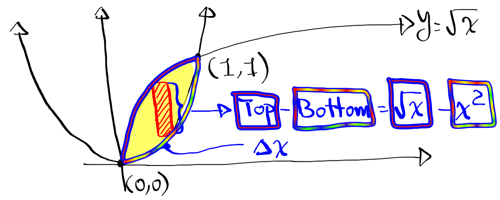
$$A = 2\int_0^2 [4 - x^2]\,dx = 2\left[4x - \frac{x^3}{3}\right]\Big|_0^2$$

$$= 2\left[4(2) - \frac{2^3}{3}\right] = 2\left[8 - \frac{8}{3}\right]$$

$$= 2 \cdot \frac{16}{3}$$

$$= \boxed{\frac{32}{3}}$$

Draw a region enclosed by $y=x^2$ and $y=\sqrt{x}$. shade it. clearly label all corner Points.



Draw a vertical reference rectangle, clearly write its length and width.

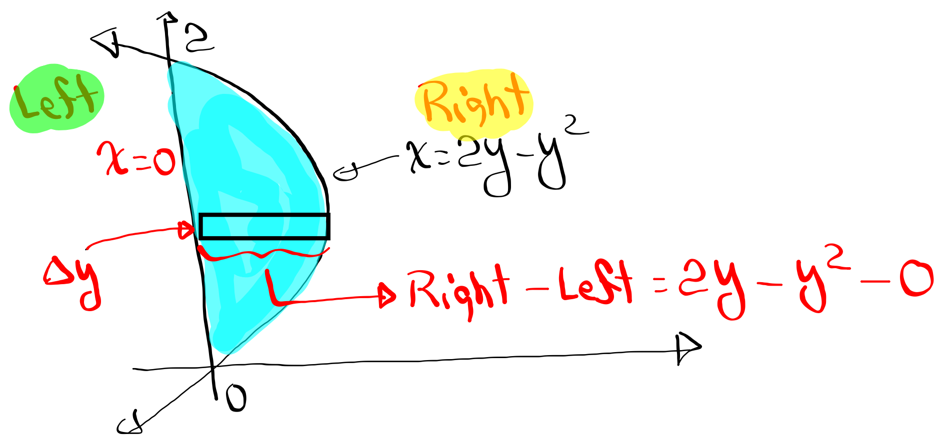
Let's find the shaded area

$$A=\int_0^1(\sqrt{x}-x^2)\,dx=\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^3}{3}\right)\Bigg|_0^1=\left[\frac{2x\sqrt{x}}{3}-\frac{x^3}{3}\right]\Bigg|_0^1$$

$$=\frac{2}{3}-\frac{1}{3}-0$$

$$=\boxed{\frac{1}{3}}$$

Find the shaded area below:



$$A=\int_0^2\left[2y-y^2-0\right]dy=\left(y^2-\frac{y^3}{3}\right)\Bigg|_0^2$$

$$=4-\frac{8}{3}-0=\boxed{\frac{4}{3}}$$

$$\int (1+x)^2\,dx$$

$$= \int (1 + 2x + x^2)\,dx = \boxed{x + x^2 + \frac{x^3}{3} + C}$$

If we let $u = 1+x \rightarrow du = dx$

$$\int (1+x)^2\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{(1+x)^3}{3} + C$$

$$= \frac{x^3 + 3x^2 + 3x + 1}{3} + C$$

$$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + C$$

$$\int (x^2-1)^2 \cdot 2x\,dx$$

$$= \int (x^4 - 2x^2 + 1)\cdot 2x\,dx$$

$$= \int (2x^5 - 4x^3 + 2x)\,dx$$

$$= \frac{2x^6}{6} - \frac{4x^4}{4} + \frac{2x^2}{2} + C$$

$$= \boxed{\frac{1}{3}x^6 - x^4 + x^2 + C}$$

Let $u = x^2 - 1$

$du = 2x\,dx$

$$\int u^2\,du =$$

$$\frac{u^3}{3} + C =$$

$$\boxed{\frac{(x^2-1)^3}{3} + C}$$

Substitution Rule

$$\int f(g(x))\, g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$ and it is differentiable.

ex: $\int x^3 \cos(x^4)\,dx$

Let $u = x^4$

$\frac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$\frac{du}{4} = x^3\,dx$

$$= \int \cos u \cdot \frac{du}{4}$$

$$= \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \boxed{\frac{1}{4}\sin x^4 + C}$$

$$\int \frac{4x}{\sqrt{4x^2-1}}\,dx$$

If $\cancel{u = 4x \to du = 4dx}$

If $u = 4x^2 - 1$

$du = 8x\,dx$

$\frac{du}{2} = 4x\,dx$

$$= \int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{1}{\cancel{2}} \cdot \frac{u^{1/2}}{\cancel{1/2}} + C = \sqrt{u} + C$$

$$= \boxed{\sqrt{4x^2-1} + C}$$

Evaluate $\int \sin x \cos x \, dx = \int u \, du$

$u = \sin x$

$du = \cos x \, dx$

$= \frac{u^2}{2} + C$

$= \boxed{\frac{\sin^2 x}{2} + C}$

Evaluate

$\int \frac{\sin x}{\cos^3 x} dx$

$\cancel{u = \sin x}$

$\cancel{du = \cos x \, dx}$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$\int \frac{-1}{u^3} du$

$= \int -u^{-3} du = -\frac{u^{-2}}{-2} + C = \frac{1}{2u^2} + C$

$= \frac{1}{2\cos^2 x} + C$

Find $\int \cos x \sec^2(\sin x) \, dx$

Let $u = \sin x$

$du = \cos x \, dx$

$\int \sec^2 u \, du = \tan u + C = \boxed{\tan(\sin x) + C}$

$$\int x(x-4)^9\,dx$$

Let $u = x-4 \Rightarrow u+4 = x$

$du = dx$

$$\int (u+4)u^9\,du = \int (u^{10} + 4u^9)\,du$$

$$= \frac{u^{11}}{11} + \frac{4u^{10}}{10} + C$$

$$= \frac{(x-4)^{11}}{11} + \frac{\overset{2}{\cancel{4}}(x-4)^{10}}{\underset{5}{\cancel{10}}} + C$$

$$\int \boxed{x^2}\sqrt{x+2}\,dx$$

$u = \sqrt{x+2}$

$u^2 = x+2 \Rightarrow u^2-2 = x$

$2u\,du = dx$

$$= \int (u^2-2)^2 \cdot u \cdot 2u\,du$$

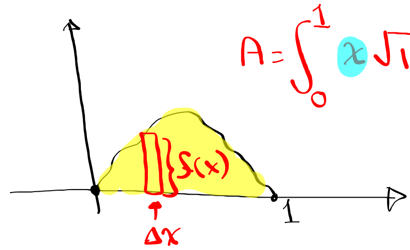
$$= \int (u^4 - 4u^2 + 4) \cdot 2u^2\,du =$$

$$= \int (2u^6 - 8u^4 + 8u^2)\,du =$$

$$\frac{2u^7}{7} - \frac{8u^5}{5} + \frac{8u^3}{3} + C =$$

$$\frac{2(\sqrt{x+2})^7}{7} - \frac{8(\sqrt{x+2})^5}{5} + \frac{8(\sqrt{x+2})^3}{3} + C$$

Find the area below $f(x) = x\sqrt{1-x^2}$, above x-axis on $[0,1]$.



$A = \int_0^1 x\sqrt{1-x^2}\,dx$

$u = 1-x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

$x=0 \rightarrow u=1$

$x=1 \rightarrow u=0$

$A = \int_1^0 \sqrt{u}\,\frac{du}{-2}$

$= \frac{-1}{2}\int_1^0 u^{1/2}\,du = \frac{-1}{\cancel{2}}\cdot \frac{u^{3/2}}{\frac{3}{\cancel{2}}}\Big|_1^0$

$= \frac{-1}{3}\,u\sqrt{u}\Big|_1^0 = \frac{-1}{3}[0-1] = \boxed{\frac{1}{3}}$

Class QZ 13

1) Evaluate $\int_0^2 (8x^3+3x^2)\,dx$

$= \left(\frac{8x^4}{4} + \frac{3x^3}{3}\right)\Big|_0^2 = (2x^4+x^3)\Big|_0^2 = 2\cdot 2^4 + 2^3 = \boxed{40}$

2) Find $\int x\sec x^2 \tan x^2\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$= \int \sec u \tan u \cdot \frac{du}{2} = \frac{1}{2}\sec u + C$

$= \boxed{\frac{1}{2}\sec x^2 + C}$